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A deformation of the $*$-product of Gutt on a reductive Lie algebra

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Some formulae for $*$-products in $\mathbb{R}^n$, which involve the exponential of the Laplacian on polynomials, are developed. In particular, integral formulae which allow us to extend the $*$-product to a space of concentrated Gaussians on $\mathbb{R}^n$ are given. Some of these results are generalized to the case of general Lie algebra, where the Baker–Campbell–Hausdorff formula is required.

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The notion of a $*$-product as a particular deformation of the associative algebra of smooth functions on a symplectic manifold was introduced in Bayen et al. (1978) to give an autonomous phase space formulation of quantum mechanics without operators. The problem of existence and construction of $*$-products on various symplectic manifolds has been studied by a number of authors [see, for example, Lichnerowicz (1981) and Lugo (1981)].

If G is a Lie group and $\mathfrak{g}^*$ the dual of the Lie algebra $\mathfrak{g}$, then the symplectic structure on any co-adjoint orbit $\theta \subseteq \mathfrak{g}^*$ is closely related to the Poisson structure on $\mathfrak{g}^*$. This suggests the study of $*$-products on $\mathfrak{g}^*$. Gutt (1983) has shown that a $*$-product $*_\lambda$ exists on any $\mathfrak{g}^*$ and has given an explicit formula for it. This structure on $S(\mathfrak{g})$ may be obtained from the universal enveloping algebra $U(\mathfrak{g})$ by a modification of the familiar symmetrization map between $S(\mathfrak{g})$ and $U(\mathfrak{g})$.

There are, however, some reasons for considering variants of the Gutt $*$-product. The symmetrization map is not entirely natural and for some applications in harmonic analysis it is preferable to use the maps of Harish-Chandra or Duflo. Also in Wildberger (1993) a canonical $*$-product was constructed on any integral co-adjoint orbit of a compact Lie group. While investigating the relationship between these orbit $*$-products and $*$-products on $\mathfrak{g}^*$, we have found that when $\mathfrak{g}$ is reductive there exists a family of $*$-products $*_{\lambda,c}$ on $\mathfrak{g}^*$ which include the Gutt $*$-products $*_\lambda$ as the special case $c=0$.

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Surprisingly, this product is non-trivial even in the case of $\mathfrak{g}=\mathbb{R}$, where it is given by the formula

$$X^n *_c X^m = \sum_{p=0}^{\min(n,m)} \binom{n}{p}\binom{m}{p} p!\, c^p X^{n+m-2p}$$

on monomials. Of particular interest is an inner product $\langle\ ,\ \rangle_{\lambda,c}$ on $S(\mathfrak{g})$ associated to these $*$-products by the formula

$$\langle f, g\rangle_{\lambda,c} = f *_{\lambda,c} g(0)$$

for f, $g \in S(\mathfrak{g})$. In the case of $\mathfrak{g}=\mathbb{R}$, similar inner products have been utilized in commutative harmonic analysis [see Coifman and Weiss (1971)].

We develop some formulae for these $*$-products in $\mathbb{R}^n$ which involve the exponential of the Laplacian on polynomials. In particular we give integral formulae which allow us to extend the $*$-product to a space of concentrated Gaussians on $\mathbb{R}^n$. We are able to generalize some of these results to the case of general $\mathfrak{g}$, where the Baker–Campbell–Hausdorff formula is required.

1. Let G be a real Lie group with Lie algebra $\mathfrak{g}$. Let $S(\mathfrak{g})$ denote the symmetric algebra over $\mathbb{C}$ and $S_{\mathbb{R}}(\mathfrak{g}) \subset S(\mathfrak{g})$ the symmetric algebra over $\mathbb{R}$. These may be viewed, respectively, as complex valued and real valued polynomials on the dual of the Lie algebra $\mathfrak{g}^*$. The Lie bracket $[\ ,\ ]$ of $\mathfrak{g}$ extends to a Poisson structure on $S(\mathfrak{g})$.

An algebra structure $*$ on $S(\mathfrak{g})$ will be called a $*$-product if

(i) it is associative,

(ii) $\exists \lambda \in \mathbb{R}$ such that for all p, $q \in S(\mathfrak{g})$,

$$p * q = pq + i\lambda[p, q] + \text{lower degree terms}\,,$$

(iii) for all p, $q \in S_{\mathbb{R}}(\mathfrak{g})$,

$$q * p = \overline{p * q}\,.$$

Gutt (1983) has defined a $*$-product structure on $S(\mathfrak{g})$, which we may write as follows. For $X, X_1, \ldots, X_k \in \mathfrak{g}$,

$$X *_\lambda (X_1 \cdots X_k) = X X_1 \cdots X_k + \sum_{r=1}^{k} \frac{(i\lambda)^r B_r}{r!} \times \sum_{\langle j_1,\ldots,j_r\rangle \subset \{1,\ldots,k\}} [\cdots[[X, X_{j_1}], X_{j_2}]\cdots X_{j_r}] X_1 \cdots \hat{X}_{j_1} \cdots \hat{X}_{j_r} \cdots X_k\,,$$

where $\langle j_1, \ldots, j_r\rangle$ signifies ordered set, $\hat{}$ denotes deletion and B_i is the ith Bernoulli number.

Now suppose that $\mathfrak{g}$ is reductive so that we may choose a real non-degenerate symmetric $\mathfrak{g}$-invariant form $(\ ,\ )$ on $\mathfrak{g}$. For each $X \subset \mathfrak{g}$ let $\partial_X : S(\mathfrak{g}) \to S(\mathfrak{g})$ be the derivation that extends

$$\partial_X(Y) = (X, Y) \quad \forall Y \in \mathfrak{g} .$$

For $X \in \mathfrak{g}$, $p \in S(\mathfrak{g})$ and λ, $c \in \mathbb{R}$, define

$$X *_{\lambda,c} p = X *_\lambda p + c\partial_X(p) .$$

Theorem 1. *$*_{\lambda,c}$ extends in a unique way to a $*$-product on $S(\mathfrak{g})$.* □

For $p, q \in S(\mathfrak{g})$, define

$$\langle p, q \rangle_{\lambda,c} = p *_{\lambda,c} q(0) .$$

Theorem 2. *$\langle\ ,\ \rangle_{\lambda,c}$ is a symmetric real valued non-degenerate bilinear form on $S(\mathfrak{g})$. It satisfies*

$$\langle p *_{\lambda,c} q, r \rangle = \langle p, q *_{\lambda,c} r \rangle$$

for all $p, q, r \in S(\mathfrak{g})$. □

2. The $*$-product defined above is non-trivial even when $\mathfrak{g}$ is abelian. Here we consider the case $\mathfrak{g} = \mathbb{R}^n$ with $(\ ,\ )$ a non-degenerate bilinear symmetric form and $\{X_1, ..., X_n\}$ an orthonormal basis. The $*$-product depends only on c, so we write it as $*_c$.

Lemma 3.

$$\begin{aligned}(X_1^{\alpha_1} \cdots X_n^{\alpha_n}) *_c (X_1^{\beta_1} \cdots X_n^{\beta_n}) \\ = \sum_{p_1=0}^{\min(\alpha_1,\beta_1)} \cdots \sum_{p_n=0}^{\min(\alpha_n,\beta_n)} \binom{\alpha_1}{p_1}\binom{\beta_1}{p_1} p_1! \cdots \binom{\alpha_n}{p_n}\binom{\beta_n}{p_n} p_n! \\ \times c^{p_1+\cdots+p_n} X_1^{\alpha_1+\beta_1-2p_1} \cdots X_n^{\alpha_n+\beta_n-2p_n} .\end{aligned}$$ □

Note that $*_c$ in this case is commutative.

Lemma 4.

$$\langle X_1^{\alpha_1} \cdots X_n^{\alpha_n}, X_1^{\beta_1} \cdots X_n^{\beta_n} \rangle_c = \delta_{\alpha_1\beta_1} \cdots \delta_{\alpha_n\beta_n} \alpha_1! \cdots \alpha_n! c^{\alpha_1+\cdots+\alpha_n} .$$ □

Therefore, if $c > 0$, then $\langle\ ,\ \rangle_c$ is positive definite.

Now for any monomial $p = X_{i_1} \cdots X_{i_k}$ define $A_c(p) \in S(\mathfrak{g})$ by

$$A_c(p) = X_{i_1} *_c \cdots *_c X_{i_k} .$$

This is a well-defined polynomial of degree k whose k component is exactly p, so that A_c extends linearly to a vector space isomorphism $A_c:S(\mathfrak{g})\to S(\mathfrak{g})$. We now describe this map explicitly. Let

$$\Delta=\partial^2/\partial X_1^2+\cdots+\partial^2/\partial X_n^2$$

be the Laplacian which we consider as an operator on $S(\mathfrak{g})$.

Proposition 5. *$A_c=\mathrm{e}^{c\Delta/2}$. That is, for any $p\in S(\mathfrak{g})$,*

$$A_c(p)=\mathrm{e}^{c\Delta/2}(p)=\sum_{n=0}^{\infty}\frac{1}{n!}(c\Delta/2)^n(p)\ ,$$

this series having only a finite number of terms. □

Corollary 6. *$p\in S(\mathfrak{g})$ is harmonic if and only if $p=A_c(p)$.* □

3. We now investigate the possibility of extending the $*$-product $*_{\lambda,c}$ from polynomials to more general functions on $\mathfrak{g}^*$. For simplicity, we first consider the case $\mathfrak{g}=\mathbb{R}$.

Lemma 7. *For $t_1,t_2\in\mathbb{R}$ the formal series for $\mathrm{e}^{\mathrm{i}t_1X}*_c\mathrm{e}^{\mathrm{i}t_2X}$ converges to $\mathrm{e}^{\mathrm{i}(t_1+t_2)X}\mathrm{e}^{-ct_1t_2}$.* □

If f, g are functions on $\mathbb{R}$ we may define $f*g$ formally as follows. Expand f, g using the Fourier transform

$$f(X)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(t)\mathrm{e}^{\mathrm{i}tX}\,\mathrm{d}t\ ,$$

$$g(X)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{g}(t)\mathrm{e}^{\mathrm{i}tX}\,\mathrm{d}t\ .$$

From the previous lemma, we get

$$f*_c g(X)=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\hat{f}(t_1)\hat{g}(t_2)\mathrm{e}^{\mathrm{i}(t_1+t_2)(X)}\mathrm{e}^{-ct_1t_2}\,\mathrm{d}t_1\,\mathrm{d}t_2\ .$$

Some manipulations allow us to rewrite this as

$$f*_c g(X)=\int_{\mathbb{R}}\int_{\mathbb{R}}f(X_1)g(X_2)B_c(X,X_1,X_2)\,\mathrm{d}X_1\,\mathrm{d}X_2\ , \qquad (3.1)$$

where $B_c(X,X_1,X_2)=(2\pi)^{-1}\mathrm{e}^{-c(X-X_1)(X-X_2)}$. The function $B_c(X,X_1,X_2)$ is called a *triple-kernel* for the algebra $*_c$ [this terminology was introduced in Wildberger (1993)]. We use this formal calculation to define $f*_c g$ by (3.1) whenever the integrals make sense. It is now natural to inquire if there is a space of functions on $\mathbb{R}$ for which (3.1) converges and defines an algebra.

Recall that the Gaussian $e^{-aX^2/2}$ is its own Fourier transform if and only if $a=1$. We will say that a Gaussian is *concentrated* if it has the form

$$e^{-(1/2+a)X^2}=\Omega_a$$

for $a>0$. Let $\mathscr{F}$ be the space of functions on $\mathbb{R}$ spanned by all concentrated Gaussians and their translates.

Proposition 8. *Under the product defined by (3.1)*

$$\Omega_a * \Omega_b = \frac{1}{\sqrt{2a+2b+4ab}}\,\Omega_{a\circ b}\,,$$

where

$$a\circ b = ab/(a+b+2ab)\,. \qquad \square$$

Using a more general formula involving the translates of two concentrated Gaussians one may show the following.

Proposition 9. *The space $\mathscr{F}$ is closed under the product defined by (3.1).* $\square$

We remark that the form of $B(X, X_1, X_2)$ immediately implies that $*_\lambda$ is translation invariant, so the same is true of the $*$-product on $S(\mathbb{R})$.

4. We now show how to generalize some of these results to the case of general $\mathfrak{g}$. Recall that if $\exp:\mathfrak{g}\to G$ is the exponential map there is a neighbourhood U_0 of 0 in $\mathfrak{g}$ on which exp is a diffeomorphism. We may also find neighbourhoods $U_0\supset U_1\supset U_2\supset\cdots$ of 0 such that for all $X, Y\in U_n$ $(n\geqslant 1)$ there exists $\psi(X, Y)\in U_{n-1}$ such that

$$\exp X \exp Y = \exp \psi(X, Y)$$

and where $\psi(X, Y)$ is given by the Baker–Campbell–Hausdorff formula [see, for example, Hochschild (1965)],

$$\psi(X, Y) = X+Y+\tfrac{1}{2}[X, Y]+\tfrac{1}{12}[Y, [Y, X]]+\tfrac{1}{12}[X, [X, Y]]+\cdots.$$

For simplicity let us suppose $\lambda=1$ so that $*_c = *_{1,c}$. Then the appropriate generalization of lemma 7 is the following.

Lemma 10. *Let $X, Y\in U_n$ so that $\psi(X, Y)=Z\in U_{n-1}$. Then*

$$e^{-c(X,X)/2}e^{iX} *_c e^{-c(Y,Y)/2}e^{iY} = e^{-c(Z,Z)/2}e^{iZ}$$

as an equality of formal power series. □

We see that we can now extend the *-product to more general functions on $\mathfrak{g}^*$. Explicitly, if f and g are functions on $\mathfrak{g}^*$ whose Fourier transforms $\hat{f}$ and $\hat{g}$ are supported in U_n then we may write

$$f *_c g = \frac{1}{2\pi} \int \int \hat{f}(Y_1)\hat{g}(Y_2) e^{iY} *_c e^{iY_2} \, dY_1 \, dY_2$$

and utilize lemma 10 to evaluate this integral. We also see, however, that the topology of G interferes in our attempt to find a uniform space of functions on which the *-product extends by an integral formula. These remarks also hold in the case $c=0$, i.e. the Gutt *-product.

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